Fourier series and Fourier transform

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Phasor representation of sinusoids [1]

Consider a pure sinusoidal quantity given by:

$$x(t) = X_m \cos(\omega t + \phi) \tag{1}$$

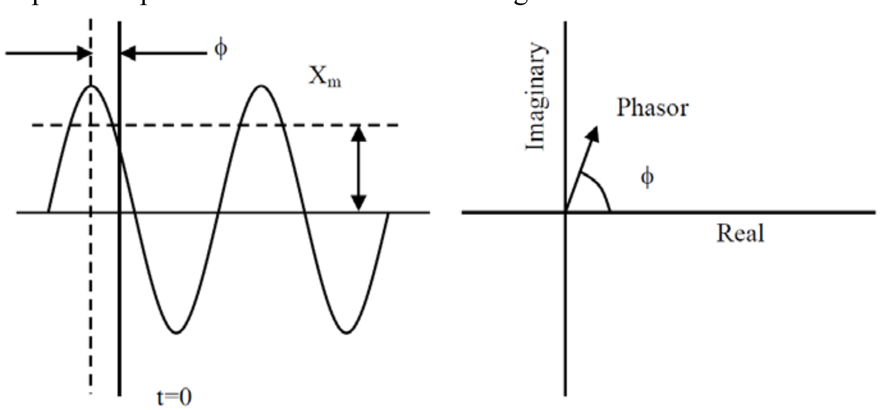
ω being the frequency of the signal in radians per second, and φ being the phase angle in radians. X_m is the peak amplitude of the signal. The *root mean square* (RMS) value of the input signal is ($X_m/\sqrt{2}$). Recall that RMS quantities are particularly useful in calculating active and reactive power in an AC circuit.

Equation (1) can also be written as:

$$x(t) = \mathrm{Re}\left\{X_m e^{j(\omega t+\phi)}\right\} = \mathrm{Re}\left[\left\{e^{j(\omega t)}\right\} X_m e^{j\phi}\right] \tag{2}$$

It is customary to suppress the term $e^{j(\omega t)}$ in the expression (1), with the understanding that the frequency is ω. The sinusoid of equation (1) is represented by a complex number X known as its *phasor representation.*

A sinusoid and its phasor representation are illustrated in Figure 1.



(a) (b)

Figure 1. A sinusoid (a) and its representation as a phasor (b). The phase angle of the phasor is arbitrary, as it depends upon the choice of the axis t = 0. Note that the length of the phasor is equal to the RMS value of the sinusoid.

Fourier series and Fourier transform

Fourier series

Let $x(t)$ be a periodic function of t, with a period equal to T. Then $x(t+kT) = x(t)$ for all integer values of k. A periodic function can be expressed as a Fourier series [1]:

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2\pi kt}{T}\right) + \sum_{k=1}^{\infty} b_k \sin\left(\frac{2\pi kt}{T}\right) \tag{3}$$

where the constants a_k and b_k are given by:

$$a_k = \frac{2}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t)\cos\left(\frac{2\pi kt}{T}\right)dt \quad k = 0,1,2,... \tag{4}$$

$$b_k = \frac{2}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t)\sin\left(\frac{2\pi kt}{T}\right)dt \quad k = 1,2,...$$

The Fourier series can also be written in the exponential form:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{\frac{j2\pi kt}{T}} \tag{5}$$

with:

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} x(t) e^{-\frac{j2\pi kt}{T}} dt \quad k = 0,\pm1,\pm2,... \tag{6}$$

Note that the summation in Eq. (5) goes from –∞ to +∞, while the summations in Eq. (3) go from 1 to +∞. The change in summation limits is accomplished by noting that the cosine and sine functions are even and odd functions of k, and thus expanding the summation limits to (–∞ to +∞) and removing the factor 2 in front of the integrals for a_k and b_k leads to the desired exponential form of the Fourier series [1].

Fourier transform

The Fourier transform of a continuous time function $x(t)$ satisfying certain integrability conditions [1] is given by

$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} dt \tag{7}$$

and the inverse Fourier transform recovers the time function from its Fourier transform:

$$x(t) = \int_{-\infty}^{+\infty} X(f) e^{j2\pi ft} df \tag{8}$$

An important function frequently used in calculations using sampled data is the impulse function $\delta(t)$ defined by

$$x(t_0) = \int_{-\infty}^{+\infty} \delta(t - t_0) x(t) dt \tag{9}$$

The convolutions of two time functions and their Fourier transforms have a convenient relationship. Consider the convolution $z(t)$ of two time functions $x(t)$ and $y(t)$ [1]:

$$z(t) = \int_{-\infty}^{+\infty} x(\tau)y(\tau - t)d\tau \equiv x(t) * y(t) \tag{10}$$

Sampled data and aliasing [1]

Sampled data from input signals are the starting point of digital signal processing. The computation of phasors of voltages and currents begins with samples of the waveform taken at uniform intervals $k\Delta T$, ($k = 0, \pm1, \pm2, \pm3, \pm4, \cdots\cdots$ }. Consider an input signal $x(t)$ which is being sampled, yielding sampled data $x(k\Delta T)$. We may view the sampled data as a time function $x'(t)$ consisting of uniformly spaced impulses, each with a magnitude $x(k\Delta T)$:

$$x'(t) = \sum_{k=-\infty}^{\infty} x(k\Delta t)\delta(t - k\Delta T) \tag{11}$$

It is interesting to determine the Fourier transform of the sampled data function given by Eq. (11).
Hence the Fourier transform $X'(f)$ of $x'(t)$ is the convolution of the Fourier transforms of $x(t)$ and of the unit impulse train.

$$X'(f) = \frac{1}{\Delta T}\sum_{k=-\infty}^{\infty} X\left(f - \frac{k}{\Delta T}\right) \tag{12}$$

The relationship between the Fourier transforms of $x(t)$ and $x'(t)$ are as shown in Figure 6. The Fourier transform of $x(t)$ is shown to be *bandlimited*, meaning that it has no components beyond a cut-off frequency f_c.
The sampled data has a Fourier transform which consists of an infinite train of the Fourier transforms of $x(t)$ centered at frequency intervals of $(k/\Delta T)$ for all k. Recall that the sampling interval is ΔT, so that the sampling frequency $f_s = (1/\Delta T)$.

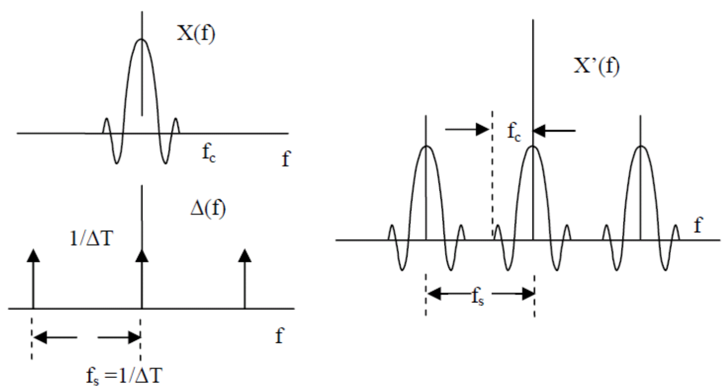


Figure 2. Fourier transform of the sampled data function as a convolution of the Transforms $X(f)$ and $\Delta(f)$. The sampling frequency is f_s, and $X(f)$ is band-limited between $\pm f_c$.

If the cut-off frequency f_c is greater than one-half of the sampling frequency f_s, the Fourier transform of the sampled data will be as shown in Figure 1. In this case, the spectrum of the sampled data is different from that of the input signal in the region where the neighboring spectra overlap as shown by the shaded region in Figure 2. This implies that frequency components estimated from the sampled data in this region will be in error, due to a phenomenon known as *aliasing*.

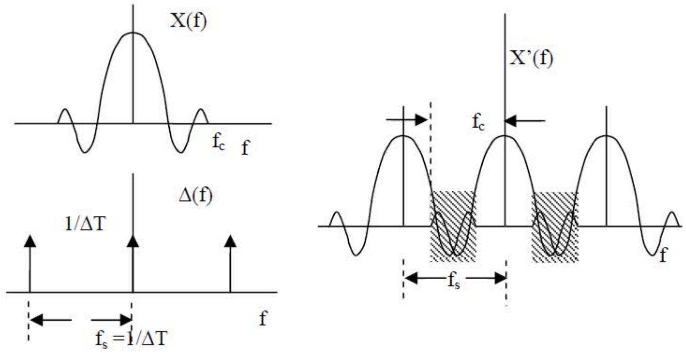


Figure 3. Fourier transform of the sampled data function when the input signal is bandlimited to a frequency greater than half the sampling frequency. The estimate of frequencies from sampled data in the shaded region will be in error because of aliasing.

It is clear from the above discussion that in order to avoid errors due to aliasing, the bandwidth of the input signal must be less than half the sampling frequency utilized in obtaining the sampled data. This requirement is known as the "*Nyquist criterion*".

Discrete Fourier transform (DFT)

Discrete Fourier Transform (DFT) is a method of calculating the Fourier transform of a small number of samples taken from an input signal $x(t)$.

The Fourier transform is calculated at discrete steps in the frequency domain, just as the input signal is sampled at discrete instants in the time domain.

Consider the collection of signal samples which fall in the data window: $x(k\Delta T)$ with $\{k = 0,1,2, \cdots ,N-1\}$. These samples can be viewed as being obtained by the multiplication of the signal $x(t)$, the sampling function $\delta(t)$, and the windowing function $\omega(t)$ [1]:

$$x'(t) = x(t)\delta(t)w(t) = \sum_{k=0}^{N-1} x(k\Delta t)\delta(t - k\Delta T) \tag{13}$$

The Fourier transform of $y(t)$ is to be sampled in the frequency domain in order to obtain the DFT of $y(t)$. The discrete steps in the frequency domain are multiples of $1/T_0$, where T_0 is the span of the windowing function. The frequency sampling function $\Phi(f)$ is given by

$$\Phi(f) = \sum_{k=-\infty}^{+\infty} \delta\left(f - \frac{n}{T_0} \right) \tag{14}$$

and the inverse Fourier transform is:

$$\phi(t) = T_0 \sum_{k=-\infty}^{+\infty} \delta(t - nT_0) \tag{15}$$

In order to obtain the samples in the frequency domain, we must multiply the Fourier transform $Y(f)$ with $\Phi(f)$. To obtain the corresponding time domain function $x'(t)$ we will require a convolution in the time domain of $y(t)$ and $\phi(t)$ [1]:

$$x'(t) = y(t) * \phi(t)$$

$$x'(t) = y(t) * \phi(t) = \left[\sum_{k=0}^{N-1} x(k\Delta t)\delta(t - k\Delta T) \right] * \left[T_0 \sum_{k=-\infty}^{+\infty} \delta(t - nT_0) \right] \tag{16}$$

$$x'(t) = y(t) * \phi(t) = T_0 \sum_{n=-\infty}^{+\infty} \left[\sum_{k=0}^{N-1} x(k\Delta t)\delta(t - k\Delta T - nT_0) \right]$$

This function is periodic with a period T_0. The functions $x(t)$, $y(t)$, and $x'(t)$ are shown in Figure 4.

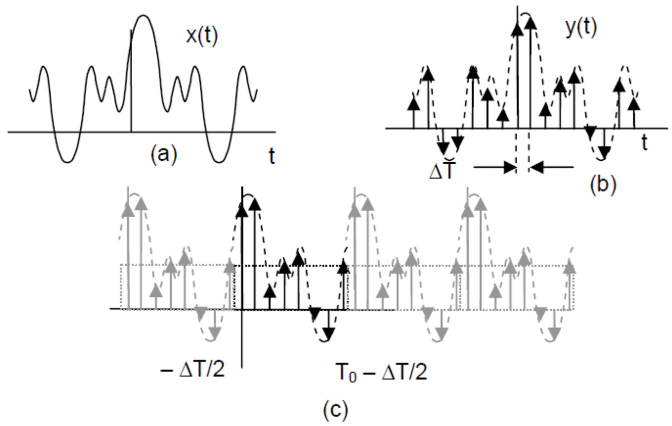


Figure 4. (a) The input function *x*(*t*), its samples (b), and (c) the Fourier transform of the windowed function *x′*(*t*).

The windowing function limits the data to samples 0 through $N - 1$, and the sampling in frequency domain transforms the original N samples in time domain to an infinite train of N samples with a period T_0 as shown in Figure 1.9 (c). Note that although the original function $x(t)$ was not periodic, the function $x'(t)$ is, and we may consider this function to be an approximation of $x(t)$.

The Fourier transform of the periodic function $x'(t)$ is a sequence of impulse functions in frequency domain of the Fourier transform [1]:

$$X'(f) = \sum_{n=-\infty}^{+\infty} \alpha_n \delta\left(f - \frac{n}{T_0} \right)$$

$$a_k = \frac{1}{T} \int_{-\frac{T_0}{2}}^{T_0 - \frac{T_0}{2}} x'(t) e^{-\frac{j2\pi nt}{T_0}} dt \qquad n = 0, \pm 1, \pm 2, \ldots \tag{17}$$

Substituting the adequate terms gives:

$$\alpha_n = \sum_{k=0}^{N-1} x(k\Delta T) e^{-\frac{j2\pi kn}{N}} \qquad n = 0, \pm 1, \pm 2, \ldots \tag{18}$$

Although the index n goes over all positive and negative integers, it should be noted that there are only N distinct coefficients α_n.

Thus, α_{N+1} is the same as α_1 and the Fourier transform $X'(f)$ has only N distinct values corresponding to frequencies $f = n/T_0$, with n ranging from 0 through $N - 1$:

$$X'\left(\frac{n}{T_0} \right) = \sum_{k=0}^{N-1} x(k\Delta T) e^{-\frac{j2\pi kn}{N}} \qquad n = 0, 1, 2, \ldots, N - 1 \tag{19}$$

Equation (19) is the definition of the DFT of N input samples taken at intervals of ΔT. The DFT is symmetric about $N/2$, the components beyond $N/2$ simply belong to negative frequency. Thus the DFT does not calculate frequency components beyond $N/(2T_0)$, which also happens to be the Nyquist limit to avoid aliasing errors.

Example of DFT calculation

Consider a periodic function:

$$x(t) = 1 + \cos(2\pi f_0 t) + \sin(2\pi f_0 t) \quad \text{(a.1)}$$

The function is already expressed in terms of its Fourier series, with $a_0 = 2$, $a_1 = 1$, and $b_1 = 1$. The signal is sampled 16 times in one period of the fundamental frequency ($f_s = Nf_0 = 960$ Hz).

```
L = 16;                    % Length of signal
Fs=L*60                    % Sample frequency
T = 1/Fs;                  % Sample time
t = (0:L-1)*T;             % Time vector
f0=60;                     % f0
```

The value of the signal is calculated per each point:

```
x = 1+cos(2*pi*f0*t) + sin(2*pi*f0*t);
```

The signal is plotted:

```
figure('Position',[   100 100 560 262])
plot(Fs*t,x,'o-k')
set(gca,'FontName','Times New Roman')
ylabel('Signal x(t)')
xlabel('time (milliseconds)')
```

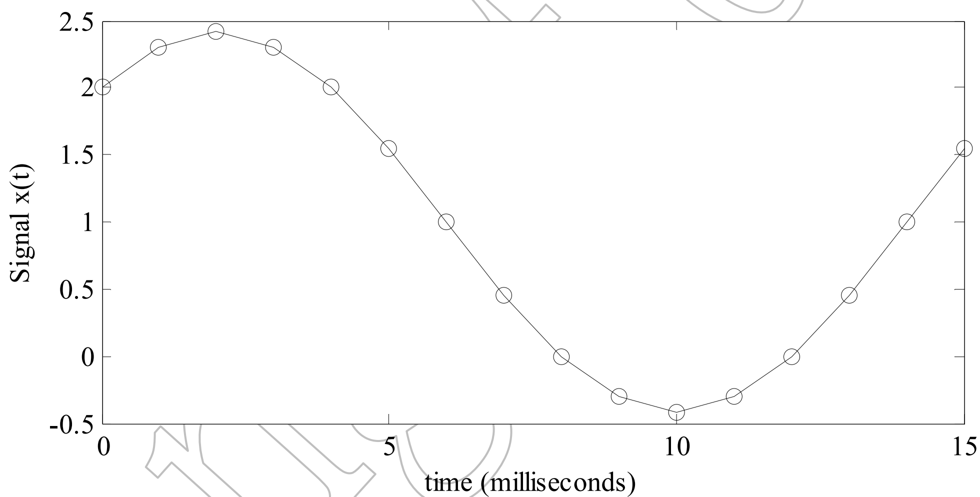


Figure 5. Sampled signal

The sampled data, the DFT, and the DFT divided by 16 (N, the number of samples) is shown in Table A.1.

Table A.1. Sampled data and Fourier transform of the periodic function $x(t) = 1 + \cos 2\pi f_0 t + \sin 2\pi f_0 t$ [1]

Sample no.	$x(t)$	Frequency	DFT	$\mathbf{X}$ = DFT/16
0	2.0000	0	16.0000	1.000
1	2.3066	f_0	$8.0000 + j8.0000$	$0.5000 + j0.5000$
2	2.4142	$2f_0$	$0.0000 - j0.0000$	$0.0000 + j0.0000$
3	2.3066	$3f_0$	$0.0000 - j0.0000$	$0.0000 + j0.0000$
4	2.0000	$4f_0$	$0.0000 + j0.0000$	$0.0000 + j0.0000$
5	1.5412	$5f_0$	$-0.0000 + j0.0000$	$0.0000 + j0.0000$
6	1.0000	$6f_0$	$0.0000 + j0.0000$	$0.0000 + j0.0000$
7	0.4588	$7f_0$	$0.0000 - j0.0000$	$0.0000 + j0.0000$
8	0.0000	–	-0.0000	$0.0000 + j0.0000$
9	–0.3066	$-7f_0$	$0.0000 + j0.0000$	$0.0000 + j0.0000$
10	–0.4142	$-6f_0$	$0.0000 - j0.0000$	$0.0000 + j0.0000$
11	–0.3066	$-5f_0$	$-0.0000 - j0.0000$	$0.0000 + j0.0000$
12	–0.0000	$-4f_0$	$0.0000 + j0.0000$	$0.0000 + j0.0000$
13	0.4588	$-3f_0$	$0.0000 + j0.0000$	$0.0000 + j0.0000$
14	1.0000	$-2f_0$	$0.0000 + j0.0000$	$0.0000 + j0.0000$
15	1.5412	$-f_0$	$8.0000 - j8.0000$	$0.5000 - j0.5000$

The last column contains the Fourier series coefficients. Note that the DC component a_0 appears in the 0th position, while the fundamental frequency component appears in the 2nd and 15th position. Using Matlab® to perform the Fast Fourier Transform (FFT) :

```
X = fft(x)
```

The DFT results are:

```
X =
  Columns 1 through 8
  16.0000   8.0000 - 8.0000i   0.0000 - 0.0000i  -0.0000   0 - 0.0000i  -0.0000   0.0000 - 0.0000i   0.0000 - 0.0000i
  Columns 9 through 16
   0.0000   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.0000   0 + 0.0000i  -0.0000   0.0000 + 0.0000i   8.0000 + 8.0000i
```

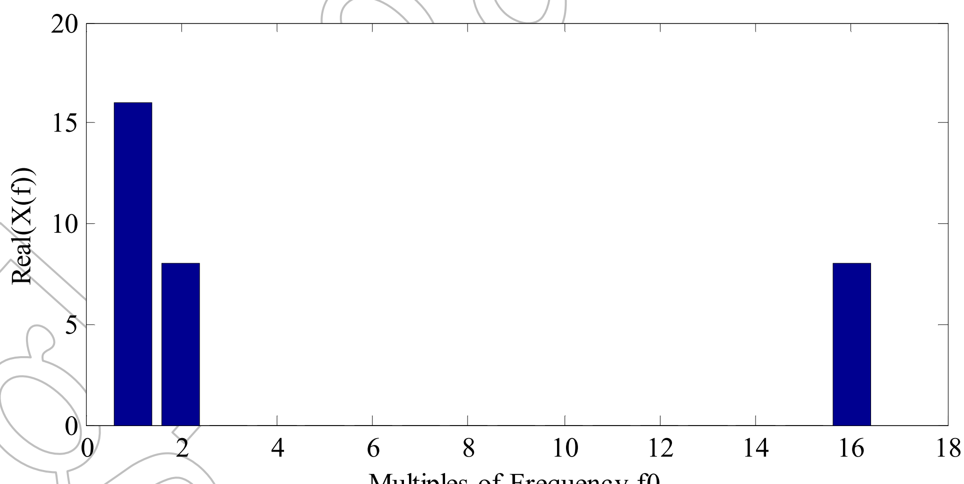


Figure 6. Real part of coefficient of DFT

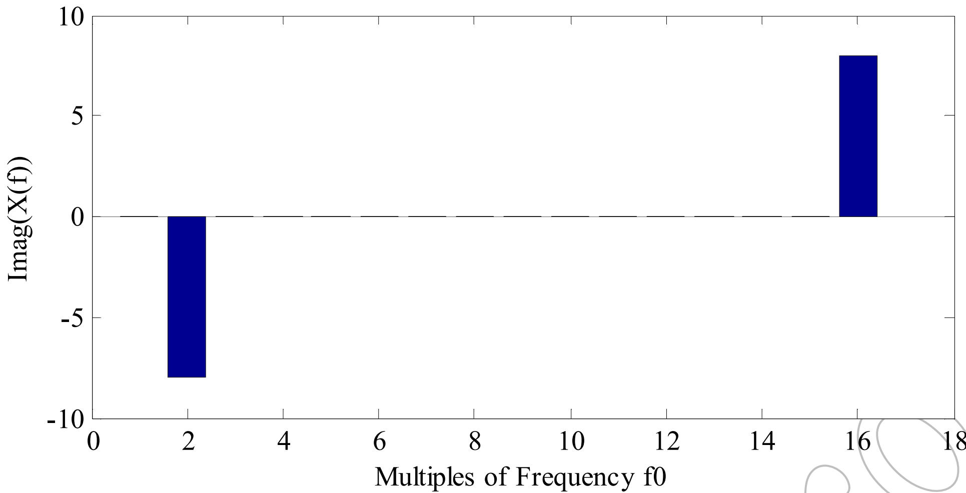


Figure 7. Imaginary part of coefficient of DFT

References

[1] A.G. Phadke,J. S. Thorp. “Synchronized Phasor Measurements and their Applications”. Springer Science+Bussines Media LLC, 2008.